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# APPLICATION OF PERTURBATION TECHNIQUES TO HEAT-TRANSFER PROBLEMS WITH VARIABLE THERMAL PROPERTIES

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**Abstract**—The paper considers the application of regular parameter perturbation technique to obtain approximate solutions of heat-transfer problems with temperature-dependent thermal properties. For pure conduction, linearly varying thermal conductivity and heat capacity are considered and two examples are solved. Next, a conducting–convecting fin is treated with convective heat-transfer coefficient proportional to $(\Delta T)^{\varepsilon}$ with $\varepsilon = 0{\cdot}25$ for fin cooled by natural convection and $\varepsilon = -0{\cdot}25$ for fin heated by laminar condensation. Finally, a two-parameter perturbation is used to solve for temperature distribution in a conducting–convecting–radiating fin with temperature-dependent thermal conductivity. On comparing the perturbation solutions with corresponding numerical solutions, the accuracy is found to be good.

## NOMENCLATURE

$A$, cross-sectional area of fin;
$C$, heat capacity;
$C_0$, heat capacity at temperature $T_0$;
erf, error function;
erfc, complementary error function;
$E$, fin parameter, $= h_b PL^2/kA$;
$E_g$, emissivity of fin surface;
$F$, nondimensional temperature, $= T/T_0$;
$h$, convective heat-transfer coefficient;
$k$, thermal conductivity;
$k_0$, thermal conductivity at zero temperature;
$L$, fin length;
$m$, a constant, equation (17);
$n$, a constant, equation (26);
$N$, fin parameter, $= (h_b PL^2/kA)^{\frac{1}{2}}$;
$P$, perimeter of fin cross-section;
$t$, time;
$T, T_0$, temperature;
$T_m$, amplitude of sinusoidal temperature variation;
$u$, function of $\eta$, equation (15);
$v$, function of $\eta$, equation (16);
$V$, transformed temperature, equation (7);
$W$, plate dimension;
$x, y$, rectangular coordinates;
$X, Y$, nondimensional coordinates.

Greek symbols

$\beta$, a constant, equation (1);
$\gamma$, a constant, equation (10);
$\theta, \phi$, nondimensional temperature;
$\eta$, similarity variable, $= x/2(kt/C_0)^{\frac{1}{2}}$;
$\sigma$, Stefan–Boltzmann constant;
$\varepsilon, \varepsilon_1, \varepsilon_2$, perturbation parameters.

Subscripts

$a$, ambient;
$s$, effective sink;
$b$, fin base;
sat, saturated condensate.

## INTRODUCTION

HEAT conduction with temperature dependent thermal properties has been studied extensively and several approximate methods of treating such problems are available [1–3]. This paper adopts yet another approach and utilises regular parameter perturbation procedure to obtain approximate solutions to heat transfer problems with either temperature dependent thermal conductivity or heat capacity or heat-transfer coefficient. For pure conduction problems, examples chosen to illustrate the procedure are two-dimensional conduction in a square plate with Dirichlet boundary conditions and linear thermal conductivity–temperature variation; and heat diffusion into a semi-infinite medium with linear heat capacity-temperature dependence. Next, a conducting–convecting straight fin is considered, and temperature distribution is obtained for the case of heat transfer coefficient proportional to $(\Delta T)^{\varepsilon}$ with $\varepsilon = 0{\cdot}25$ or $0{\cdot}33$ for fin cooled by natural convection and $\varepsilon = -0{\cdot}25$ for fin cooled by film boiling or heated by laminar condensation.

The paper also adopts two-parameter perturbation to treat problems with two independent nonlinearities which for example arise for the case of simultaneous variation of two thermal properties, or when radiation interaction with one variable thermal property is invoked. The procedure is applied to obtain second-order expansion solution for temperature distribution in a conducting–convecting–radiating fin with temperature dependent thermal conductivity.

The perturbation solutions are compared with numerical solutions and solutions based on other methods and found to be sufficiently accurate.

## PERTURBATION PROCEDURE AND EXAMPLES

In the examples that follow the mathematical details have been kept to a minimum but further details are available elsewhere [4].

### (a) *Two-dimensional, steady conduction in a square plate with variable thermal conductivity*

For two-dimensional, steady conduction in a square plate with specified boundary temperatures (Fig. 1)

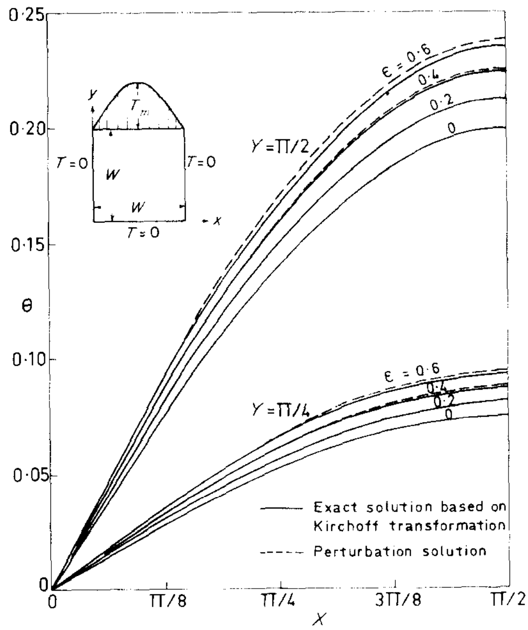


FIG. 1. Temperature distribution in a square plate with temperature-dependent thermal conductivity.

and thermal conductivity–temperature variation of the form

$$k = k_0(1+\beta T) \tag{1}$$

the Laplace equation may be written as

$$(1+\varepsilon\theta)\left(\frac{\partial^2\theta}{\partial X^2}+\frac{\partial^2\theta}{\partial Y^2}\right)+\varepsilon\left[\left(\frac{\partial\theta}{\partial X}\right)^2+\left(\frac{\partial\theta}{\partial Y}\right)^2\right] = 0 \tag{2}$$

with boundary conditions

$$\begin{aligned}\theta(0, Y) = 0, \quad \theta(\pi, Y) = 0,\\ \theta(X, 0) = 0, \quad \theta(X, \pi) = \sin X\end{aligned} \tag{3}$$

where

$$\theta = T/T_m, \quad X = \pi x/W, \quad Y = \pi y/W, \quad \varepsilon = \beta T_m. \tag{4}$$

For most materials and applications, $\varepsilon$ is small and a regular first order asymptotic expansion for $\theta$ in the perturbation parameter $\varepsilon$ may be assumed as

$$\theta = \theta_0 + \varepsilon\theta_1 + O(\varepsilon^2). \tag{5}$$

Substituting (5) into (2) and comparing coefficients of zero and unit powers of $\varepsilon$, equations for $\theta_0$ and $\theta_1$ with appropriate boundary conditions can be obtained. The complete first order solution is

$$\theta = \frac{\sinh Y \sin X}{\sinh \pi} + \varepsilon \sum_{n=1,3,\ldots,\infty} \frac{2}{n(n^2-4)\pi \sinh^2 \pi} \times \left[(1-\cosh 2\pi)\frac{\sinh nY}{\sinh n\pi} - (1-\cos 2Y)\right]\sin nX. \tag{6}$$

Equations (2, 3), however, admit exact solution. By introducing the Kirchoff transformation

$$V = \int_0^\theta k\,\mathrm{d}\theta = k_0(\theta + \tfrac{1}{2}\varepsilon\theta^2) \tag{7}$$

into equations (2, 3) one can obtain a linear problem in $V$. Employing the method of separation of variables, the solution for $V$ is

$$V = k_0\left[\frac{\sinh Y \sin X}{\sinh \pi} + \varepsilon\frac{1}{\pi}\sum_{n=1,3,\ldots,\infty} \times \left(\frac{1}{n} - \frac{n}{n^2-4}\right)\frac{\sinh nY \sin nX}{\sinh n\pi}\right] \tag{8}$$

and the solution for $\theta$ is

$$\theta = \frac{1}{\varepsilon}\left[\left(1+\frac{2\varepsilon V}{k_0}\right)^{\frac{1}{2}} - 1\right]. \tag{9}$$

The perturbation temperature distributions at $Y = \pi/4$ and $Y = \pi/2$ are compared with exact solutions in Fig. 1. Excellent agreement is obtained even for $\varepsilon$ as large as 0·6. Although results are shown for positive values of $\varepsilon$, the same accuracy is obtained for negative values of $\varepsilon$.

### (b) *Transient heat diffusion into a semi-infinite medium with variable heat capacity*

The medium is assumed at temperature $T = 0$ for time $t < 0$ and its surface at $x = 0$ is suddenly changed to $T_0$ at $t = 0$. The thermal conductivity is taken constant but the heat capacity is assumed to be of the form

$$C = C_0[1+\gamma(T-T_0)]. \tag{10}$$

The similarity equation for temperature distribution together with boundary conditions is

$$F'' + 2\eta[1+\varepsilon(F-1)]F' = 0 \tag{11}$$

$$\eta = 0,\ F = 1; \qquad \eta = \infty,\ F = 0 \tag{12}$$

where

$$F = \frac{T}{T_0}, \quad \eta = \frac{x}{2\left(\dfrac{kt}{C_0}\right)^{\frac{1}{2}}}, \quad \varepsilon = \gamma T_0 \tag{13}$$

and primes denote differentiation with respect to $\eta$.

As $\varepsilon$ is usually small, the solution for $F$ can be assumed of the form of equation (5). Proceeding as before, the solution for zero-order problem $F_0$ is found to be erfc $\eta$, which practically decays to zero for $\eta \geqslant 3$. To effect the solution for the first-order problem $F_1$ in closed form, the complementary error function representing $F_0$ is replaced by a third-order least squares polynomial (accuracy $\pm 0{\cdot}5$ per cent) in the range $0 \leqslant \eta \leqslant 3$ and so is the boundary $\eta = \infty$ by $\eta = 3$. The two-term perturbation solution thus obtained is

$$F = \operatorname{erfc}\eta + \varepsilon[u + (v - 0{\cdot}3272)\operatorname{erf}\eta] \qquad (14)$$

where

$$u = -0{\cdot}8708\eta^4 + 0{\cdot}6000\eta^5 - 0{\cdot}1780\eta^6 + 0{\cdot}0259\eta^7 - 0{\cdot}0016\eta^8 \qquad (15)$$

$$v = 0{\cdot}8798\eta^3 - 0{\cdot}2842\eta^4 + 0{\cdot}0319\eta^5. \qquad (16)$$

The perturbation solution is plotted in Fig. 2 for $\varepsilon = 0{\cdot}5$ together with solutions using optimal linearisation [3] and variational method [5] and compared with the numerical solution based on Runge–Kutta method.

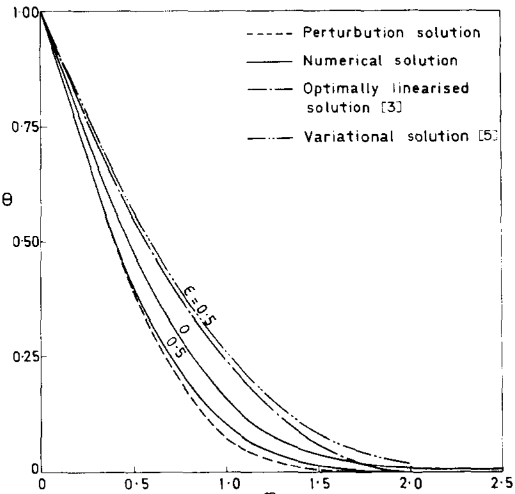


FIG. 2. Temperature distribution in a semi-infinite solid with temperature dependent heat capacity.

The perturbation solution is very close to numerical solution but the linearised and variational solutions are significantly in error. This is due to the fact that the latter both employ quadratic approximation for $F_0$ in generating the solution for $F$. This is rather poor whereas the third order polynomial utilised in the present work is a very accurate representation of $F_0$.

(c) *Conducting–convecting fin with variable heat-transfer coefficient*

A straight fin of length $L$, cross-sectional area $A$ and perimeter $P$ exposed on both sides to free convective environment at temperature $T_a$ is considered. The commonly used boundary conditions of constant base temperature $T_b$ and adiabatic tip are assumed. The thermal conductivity of the fin is taken constant while the convective heat-transfer coefficient $h$ is assumed to be of the form

$$h = m(T - T_a)^{\varepsilon} \qquad (17)$$

where $m$ is a constant and $\varepsilon = 0{\cdot}25$ and $0{\cdot}33$ for laminar and turbulent conditions respectively. Equation (17) is applicable to a horizontal fin but if the fin is vertical, $h$ also depends on coordinate $x$. Measuring the axial distance from fin tip and using one-dimensional approximation, equations governing the temperature distribution can be written as

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}X^2} - N^2\theta^{1+\varepsilon} = 0 \qquad (18)$$

$$X = 0,\ \frac{\mathrm{d}\theta}{\mathrm{d}X} = 0; \quad X = 1,\ \theta = 1 \qquad (19)$$

where

$$\theta = \frac{T - T_a}{T_b - T_a}, \quad X = \frac{x}{L}, \quad N^2 = \frac{h_b P L^2}{kA}, \quad h_b = m(T_b - T_a)^{\varepsilon}. \qquad (20)$$

Since $\varepsilon$ is small, $\theta$ is assumed of the form of equation (5). Further, expressing $(\theta_0 + \varepsilon\theta_1)^{\varepsilon}$ as $\exp[\varepsilon \ln(\theta_0 + \varepsilon\theta_1)]$ it can be shown by expansion that

$$(\theta_0 + \varepsilon\theta_1)^{\varepsilon} = 1 + \varepsilon \ln\theta_0 + \varepsilon^2\left[\frac{\theta_1}{\theta_0} + \tfrac{1}{2}(\ln\theta_0)^2\right] + O(\varepsilon^3). \qquad (21)$$

Proceeding as before and utilising (21), the solution for zero-order problem $\theta_0$ is found to be

$$\theta_0 = \operatorname{sech} N \cosh NX. \qquad (22)$$

It is found that the equation for first-order problem $\theta_1$ contains a nonhomogeneous term of the form $N^2\theta_0 \ln\theta_0$ which according to (22) becomes

$$N^2 \operatorname{sech} N(\cosh NX \ln \operatorname{sech} N + \cosh NX \ln \cosh NX)$$

in which the presence of the second term precludes exact solution for $\theta_1$. In practice, $N$ does not exceed 2 (optimum $N = 1{\cdot}4192$ for minimum mass fin) and since $0 \leqslant X \leqslant 1$, the operating range of the fin is such that $0 \leqslant NX \leqslant 2$. For this range, the function $\cosh NX \ln \cosh NX$ is approximated quite accurately by its eight-order truncated Maclaurin's series. With this approximation, the solution for $\theta_1$ is obtained. Thus, the complete first order solution becomes

$$\theta = \operatorname{sech} N \cosh NX + \varepsilon\{C_1 \cosh NX + C_2 X \sinh NX - C_3[84 + 42(NX)^2 + \tfrac{41}{12}(NX)^4 + \tfrac{37}{360}(NX)^6 + \tfrac{1}{560}(NX)^8]\} \qquad (23)$$

where

$$C_1 = \tfrac{1}{2}\operatorname{sech}^2 N(84 + 42N^2 + \tfrac{41}{12}N^4 + \tfrac{37}{360}N^6 + \tfrac{1}{560}N^8) - \tfrac{1}{2}N \tanh N \operatorname{sech} N \ln \operatorname{sech} N \qquad (24)$$

$$C_2 = \tfrac{1}{2}N \operatorname{sech} N \ln \operatorname{sech} N, \quad C_3 = \tfrac{1}{2}\operatorname{sech} N. \qquad (25)$$

The perturbation solution is displayed in Fig. 3 for $\varepsilon = 0{\cdot}25$ and $0{\cdot}33$ curves $A$ being for $N = 1{\cdot}0$ and curves $B$ being for $N = 2{\cdot}0$. Comparison with the numerical solutions indicates high accuracy of the perturbation solution.

Consideration is now given to the case when the fin is cooled by film boiling or heated by laminar condensation. A horizontal fin of cross-sectional area $A$, perimeter $P$ and length $L$ heated by laminar condensation is considered. The heat-transfer coefficient is of the form

$$h = n(T_{\mathrm{sat}} - T)^{-0{\cdot}25} \qquad (26)$$

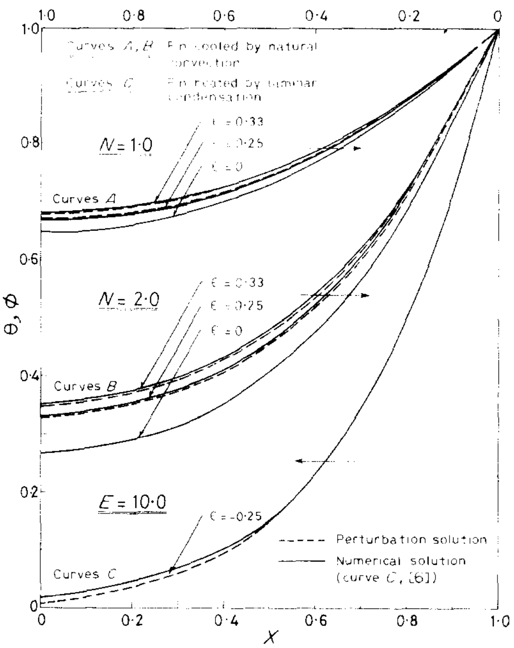


FIG. 3. Temperature distribution in a conducting–convecting fin with temperature dependent heat-transfer coefficient.

where $T_{sat}$ is saturated condensate temperature and $n$ is a constant which can be determined using Nusselt–Rohsenow theory [6]. The equation governing the temperature distribution in the fin is

$$\frac{d^2\phi}{dX^2} - E\phi^{\frac{3}{4}} = 0 \tag{27}$$

where

$$\phi = \frac{T_{sat} - T}{T_{sat} - T_b}, \quad X = \frac{x}{L}, \quad E = \frac{h_b P L^2}{kA}$$
$$h_b = n(T_{sat} - T_b)^{-0\cdot 25} \tag{28}$$

and the coordinate $x$ is measured from the base of the fin.

Lienhard and Dhir [6] have recently considered this problem and shown that for good fin design $E = O(10)$ which gives a tip temperature of about 0·03 for adiabatic tip condition. Hence, the boundary condition at the tip may be approximated without much loss of accuracy by that of infinitely long fin. Equation (27) is solved subject to the boundary conditions

$$X = 0, \ \phi = 1; \quad X = \infty, \ \phi = 0. \tag{29}$$

With $E = N^2$ and $\varepsilon = -0{\cdot}25$, equation (18) is identical to equation (27). Following the procedure adopted previously, the two-term perturbation solution of equations (27 and 29) can be easily obtained as

$$\phi = [1 - \tfrac{1}{16}E^{\frac{1}{2}}(X + E^{\frac{1}{2}}X^2)]\exp(-E^{\frac{1}{2}}X). \tag{30}$$

The perturbation solution and numerical solution [6] are shown as curves $C$ in Fig. 3 and agree very closely.

### (d) *Conducting–convecting–radiating fin with variable thermal conductivity*

This example demonstrates the application of two-parameter perturbation to a problem involving two nonlinearities. Consider one-dimensional conduction in a straight fin of length $L$, cross-sectional area $A$ and perimeter $P$, the surface heat transfer involving simultaneous convection and radiation. Let the convective environment temperature and effective sink temperature be $T_a$ and $T_s$. Assuming the convective heat-transfer coefficient $h$ and surface emissivity $E_g$ to be constant but allowing the thermal conductivity to vary according to

$$k = k_a[1 + \beta(T - T_a)] \tag{31}$$

the energy equation and boundary conditions (constant base temperature $T_b$ and adiabatic tip) may be written as

$$[1 + \varepsilon_1(\theta - \theta_a)]\frac{d^2\theta}{dX^2} + \varepsilon_1\left(\frac{d\theta}{dX}\right)^2 - N^2(\theta - \theta_a) - \varepsilon_2(\theta^4 - \theta_s^4) = 0 \tag{32}$$

$$X = 0, \ \frac{d\theta}{dX} = 0; \quad X = 1, \ \theta = 1 \tag{33}$$

where

$$\theta = \frac{T}{T_b}, \quad \theta_a = \frac{T}{T_a}, \quad \theta_s = \frac{T_s}{T_b}$$
$$X = \frac{x}{L}, \quad N^2 = \frac{hPL^2}{k_a A}, \quad \varepsilon_1 = \beta T_b \tag{34}$$
$$\varepsilon_2 = \frac{E_g \sigma T_b^3 P L^2}{k_a A} \text{ (radiation–conduction parameter).}$$

As mentioned earlier, $\varepsilon_1$ is usually small and for the case of weak radiation–conduction interaction, $\varepsilon_2$ is small. Therefore, an asymptotic expansion for $\theta$ in $\varepsilon_1$ and $\varepsilon_2$ may be assumed as

$$\theta = \sum_{j=0}^{\infty}\sum_{i=0}^{j} \varepsilon_1^{j-i}\varepsilon_2^{i}\theta_{i,j-i}. \tag{35}$$

The case of $\theta_a = \theta_s = 0$ is considered. The solution for the more general case can be obtained in a similar manner though the algebra is somewhat lengthy. Assuming second order expansion,

$$\theta = \theta_{00} + \varepsilon_1\theta_{01} + \varepsilon_2\theta_{10} + \varepsilon_1^2\theta_{02} + \varepsilon_1\varepsilon_2\theta_{11} + \varepsilon_2^2\theta_{20}. \tag{36}$$

Substituting (36) into (32) and (33), taking $\theta_a = \theta_s = 0$ and equating the appropriate coefficients, boundary value problems for $\theta_{00}, \theta_{01}, \ldots \theta_{20}$ can be obtained. For brevity, only the solutions are presented. These are

$$\theta_{00} = \operatorname{sech} N \cosh NX \tag{37}$$

$$\theta_{01} = \tfrac{1}{3}\operatorname{sech}^2 N[(\cosh 2N \operatorname{sech} N \cosh NX - \cosh 2NX)] \tag{38}$$

$$\theta_{10} = \tfrac{3}{8}\frac{\operatorname{sech}^4 N}{N^2}[(1 - \tfrac{4}{9}\cosh 2N - \tfrac{1}{45}\cosh 4N)\operatorname{sech} N \cosh NX + \tfrac{4}{9}\cosh 2NX + \tfrac{1}{45}\cosh 4NX - 1] \tag{39}$$

$$\theta_{02} = \tfrac{1}{6}\operatorname{sech}^3 N[(\tfrac{4}{3}\operatorname{sech}^2 N \cosh^2 2N - \tfrac{9}{8}\operatorname{sech} N \cosh 3N - \tfrac{1}{2}N \tanh N)\cosh NX - \tfrac{4}{3}\operatorname{sech} N \cosh 2N \cosh 2NX + \tfrac{9}{8}\cosh 3NX + \tfrac{1}{2}NX \sinh NX] \tag{40}$$

$$\theta_{11} = \tfrac{1}{2}\frac{\operatorname{sech}^5 N}{N^2}\{-\cosh 2N \operatorname{sech} N + [\tfrac{1}{2}(3-\tfrac{4}{3}\cosh 2N - \tfrac{1}{15}\cosh 4N)\operatorname{sech} N \cosh 2N + \tfrac{103}{320}\cosh 3N + \tfrac{13}{576}\cosh 5N + \tfrac{9}{24}N\sinh N]\operatorname{sech} N \cosh NX - \tfrac{1}{2}(1-\tfrac{4}{3}\cosh 2N - \tfrac{1}{45}\cosh 4N)\operatorname{sech} N \cosh 2NX - \tfrac{103}{320}\cosh 3NX + \tfrac{1}{45}\operatorname{sech} N \cosh 2N \cosh 4NX - \tfrac{13}{576}\cosh 5NX - \tfrac{9}{24}NX\sinh NX\} \quad (41)$$

$$\theta_{20} = \tfrac{3}{4}\frac{\operatorname{sech}^7 N}{N^4}\{(\tfrac{1}{3}\cosh 2N + \tfrac{1}{60}\cosh 4N - \tfrac{3}{4})\operatorname{sech} N + [(\tfrac{3}{4} - \tfrac{2}{3}\cosh 2N - \tfrac{1}{30}\cosh 4N + \tfrac{2}{135}\cosh 2N \cosh 4N + \tfrac{4}{27}\cosh^2 2N + \tfrac{1}{2700}\cosh^2 4N)\operatorname{sech} N + \tfrac{3}{160}\cosh 3N - \tfrac{23}{4320}\cosh 5N - \tfrac{1}{8640}\cosh 7N + \tfrac{21}{40}N\sinh N \operatorname{sech} N]\cosh NX - (\tfrac{4}{27}\cosh 2N + \tfrac{1}{135}\cosh 4N - \tfrac{1}{3})\operatorname{sech} N \cosh 2NX - \tfrac{3}{160}\cosh 3NX - (\tfrac{1}{135}\cosh 2N + \tfrac{1}{2700}\cosh\ N - \tfrac{1}{60}) \times \operatorname{sech} N \cosh 4NX + \tfrac{23}{4320}\cosh 5NX + \tfrac{1}{8640} \times \cosh 7NX - \tfrac{21}{40}NX\sinh NX\}. \quad (42)$$

In Fig. 4 the second order perturbation solutions for $\varepsilon_2 = 0{\cdot}2$ and $0{\cdot}6$ for various values of parameter $\varepsilon_1$ are compared with the corresponding numerical solutions, the parameter $N$ being fixed at unity. For $\varepsilon_2 = 0{\cdot}2$, the perturbation solutions match almost exactly with numerical solutions except for slight deviation for $\varepsilon_1 = 0{\cdot}6$. As $\varepsilon_2$ increases, the accuracy of the perturbation solutions decreases particularly at high values of $\varepsilon_1$. But even at $\varepsilon_1 = 0{\cdot}6$ where the maximum discrepancy occurs, the perturbation tip temperature is still within 2 per cent of the numerical solution. Further calculations show that the accuracy of the perturbation solution increases as $N$ increases and decreases as $N$ is reduced. Sparrow and Niewerth [7] have found that for optimum operating conditions, $\varepsilon_2$ ($N_r$ in the notation of [7]) varies from 0·7 to 0 as $N$ ($\sqrt{N_{cv}}$ in the notation of [7]) varies from 0 to 1·4192. Hence, it is obvious that the perturbation solution would yield accurate temperature distribution (and hence heat flux and fin efficiency) over the major portion of the range of optimum operating conditions.

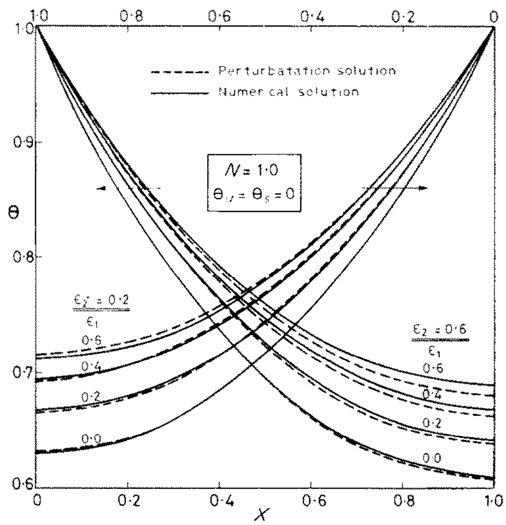


FIG. 4. Temperature distribution in a conducting–convecting–radiating fin with temperature dependent thermal conductivity.

## CONCLUSIONS

The solutions for the first two sample problems have shown that regular one-parameter perturbation technique can be usefully employed to treat conduction problems with either temperature-dependent thermal conductivity or heat capacity. The accuracy of the perturbation solutions compared with numerical solutions was excellent. The technique also yielded very accurate results when applied to fin cooled by natural convection or heated by condensation with heat-transfer coefficient proportional to fractional power of temperature difference. For problems with simultaneous variation of two thermal properties or problems involving radiation coupled with one variable thermal property, accurate solutions can be obtained using the method of two-parameter perturbation. The solution for temperature distribution in a conducting–convecting–radiating fin with temperature-dependent thermal conductivity has confirmed the usefulness of this approach.

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# APPLICATION DES TECHNIQUES DE PERTURBATIONS AUX PROBLEMES DE TRANSFERT THERMIQUE AVEC PROPRIETES THERMIQUES VARIABLES

**Résumé**—L'article traite de l'application d'une technique de perturbation à paramètre régulier pour l'obtention de solutions approchées de problèmes de transfert thermique avec propriétés thermiques dépendant de la température. Dans le cas de la conduction pure, on a considéré une conductivité thermique et une capacité calorifique variant linéairement et deux exemples sont résolus. Ensuite, on a traité le cas d'une ailette en conduction-convection avec un coefficient de transfert par convection proportionnel à $(\Delta T)^{\varepsilon}$ où $\varepsilon = 0{,}25$ lorsque l'ailette est refroidie par convection naturelle et $\varepsilon = 0{,}25$ lorsque l'ailette est chauffée par condensation laminaire. Enfin, une perturbation à deux paramètres est utilisée pour trouver la distribution de température dans une ailette à la fois conductrice, convectrice et rayonnante, avec une conductivité thermique dépendant de la température. Une comparaison des solutions de perturbation avec d'autres solution numériques montre un bon accord.

## ANWENDUNG DER STÖR-METHODE AUF WÄRMEÜBERGANGSPROBLEME MIT VERÄNDERLICHEN THERMISCHEN STOFFWERTEN

**Zusammenfassung**—Für Näherungslösungen von Wärmeübergangsproblemen mit temperaturabhängigen Stoffwerten wird die Anwendbarkeit der gewöhnlichen Parameter-Störmethode untersucht. Für Probleme der reinen Wärmeleitung wurden linear-veränderliche thermische Leitfähigkeit und Wärmekapazität herangezogen und zwei Beispiele gelöst. Weiterhin wurde ein Leitungs-Konvektionsproblem behandelt mit einem konvektiven Wärmeübergangskoeffizienten proportional $(\Delta T)^{\varepsilon}$ mit $\varepsilon = 0{,}25$ für Rippenkühlung durch natürliche Konvektion und $\varepsilon = -0{,}25$ für Rippenheizung durch laminare Kondensation. Schliesslich wurde die Zweiparameter-Störmethode verwendet für die Lösung der Temperaturverteilung in einem Rippenproblem mit Leitung, Konvektion und Strahlung bei temperaturabhängiger Wärmeleitfähigkeit. Der Vergleich der Lösungen nach der Störmethode mit entsprechenden numerischen Lösungen zeigt gute Übereinstimmung.

## ПРИМЕНЕНИЕ МЕТОДА ВОЗМУЩЕНИЙ ДЛЯ РЕШЕНИЯ ЗАДАЧ ТЕПЛОПРОВОДНОСТИ С ПЕРЕМЕННЫМИ ТЕПЛОВЫМИ СВОЙСТВАМИ

**Аннотация** — С помощью метода регулярного возмущения параметра получены приближенные решения задач теплопроводности с зависимыми от температуры тепловыми свойствами. Для случая чистой теплопроводности рассмотрены линейно изменяющиеся коэффициенты теплопроводности и теплоемкости. Численно решены два примера. Далее, рассмотрен кондуктивно-конвективный теплообмен на поверхности ребра, когда коэффициент конвективного теплообмена пропорционален $(\Delta T)^{\varepsilon}$ при $\varepsilon = 0{,}25$ для ребра, охлаждаемого естественной конвекцией, и при $\varepsilon = -0{,}25$ для ребра, нагреваемого ламинарной конденсацией. В конце используется двухпараметрический метод возмущений для определения температурного распределения в ребре в случае конвективно-кондуктивно-радиационного теплообмена, когда коэффициент теплопроводности зависит от температуры. Решения, полученные методом возмущений, хорошо согласуются с соответствующими численными решениями.